Adaptive 2D+t Space-Time Mesh Generation

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Workshop on Triangulations and More General Meshes for Space-Time Applications
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Contents

Time-dependent Navier-Stokes equations

Time dependent incompressible Navier-Stokes

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\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p - 2\nu\nabla \cdot \varepsilon(u) = f \qquad & \text{in } \Omega(t)\,, \\
\nabla \cdot u = 0 \qquad & \text{in } \Omega(t)\,, \\
+ BC + IC &
\end{aligned}
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The domain is time-dependent $\Omega(t), t \in [t_0, t_N]$

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Space-time method

Space-time algorithm

Space-time

Start with $i = 0$ and the initial condition $u(0, x) = u_0(x)$

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Space-time algorithm

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Start with $i = 0$ and the initial condition $u(0, x) = u_0(x)$

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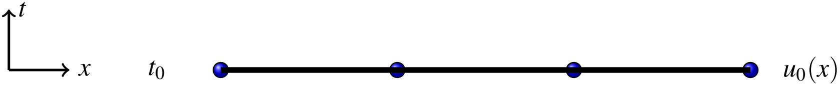
All we need is u from the previous time slab

Upwind in time



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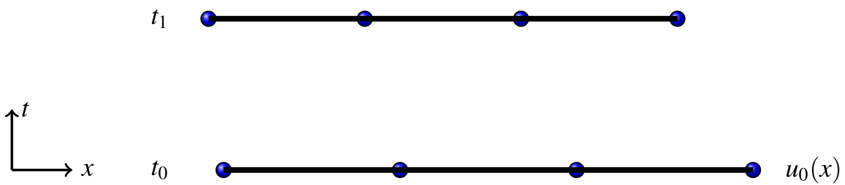

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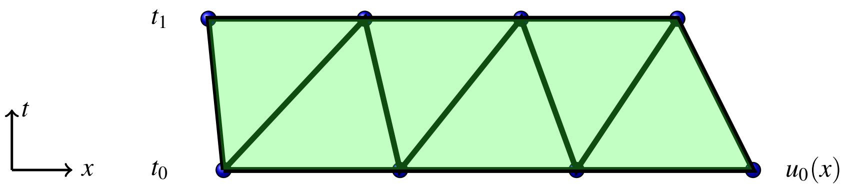

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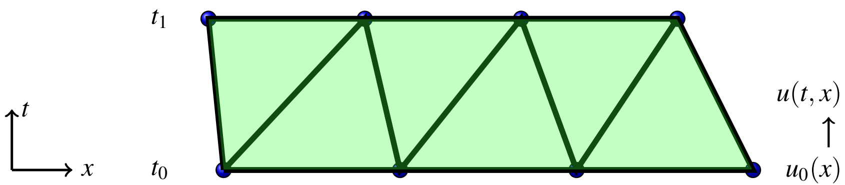

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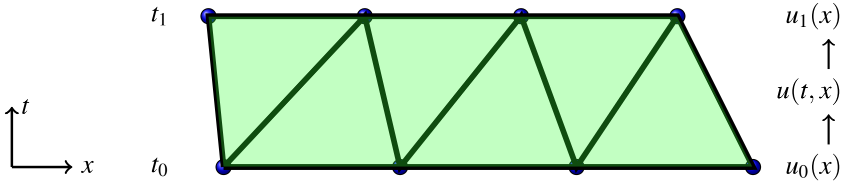

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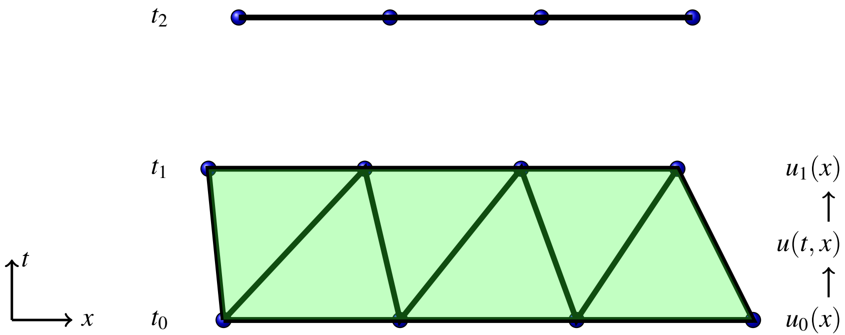

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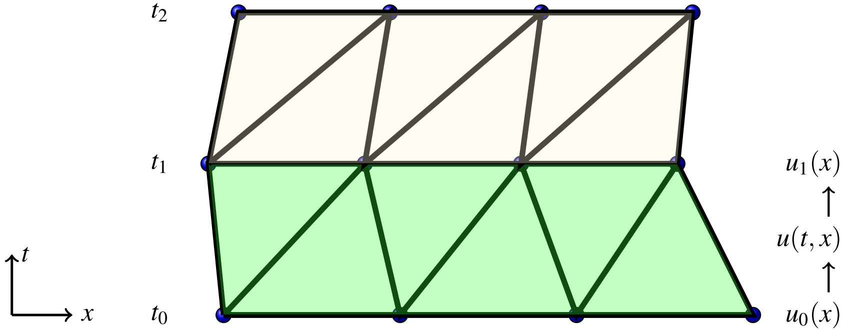

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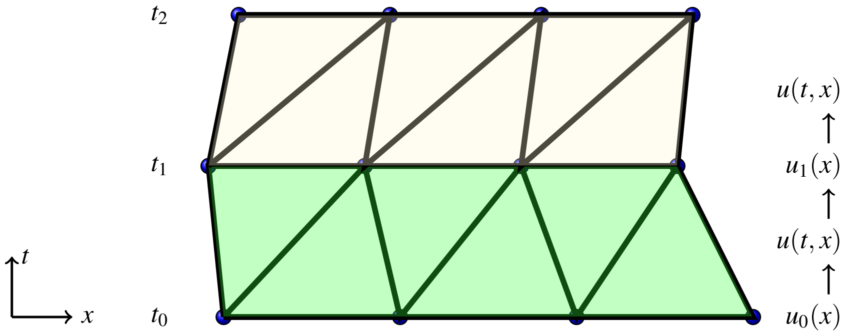

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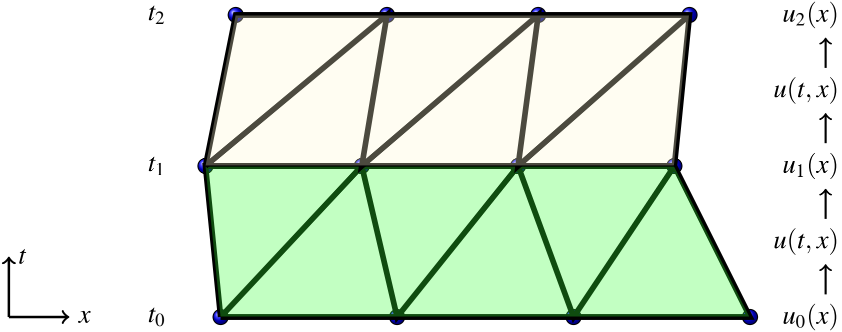

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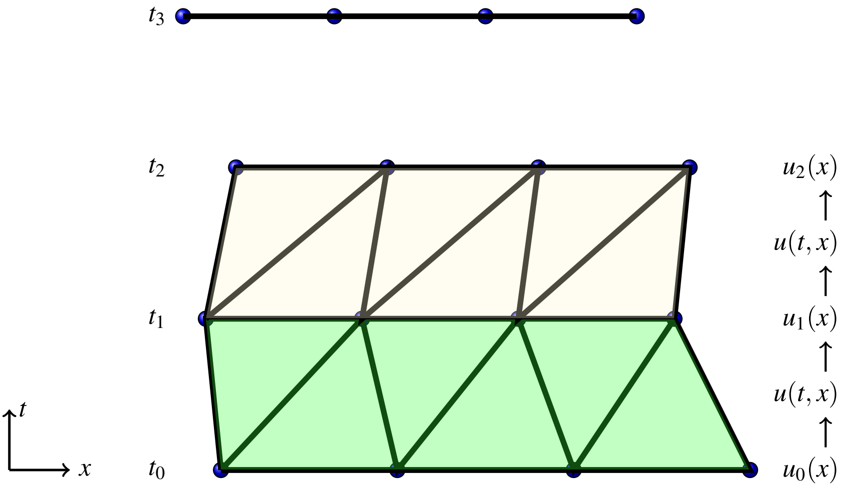

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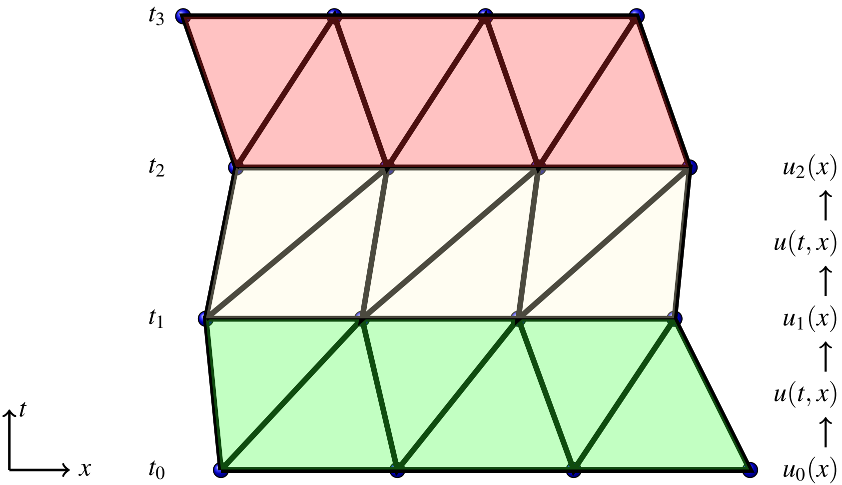

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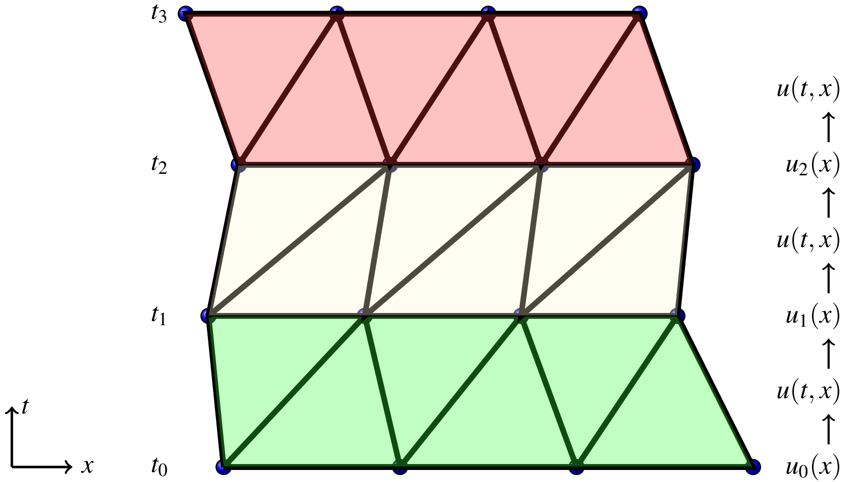

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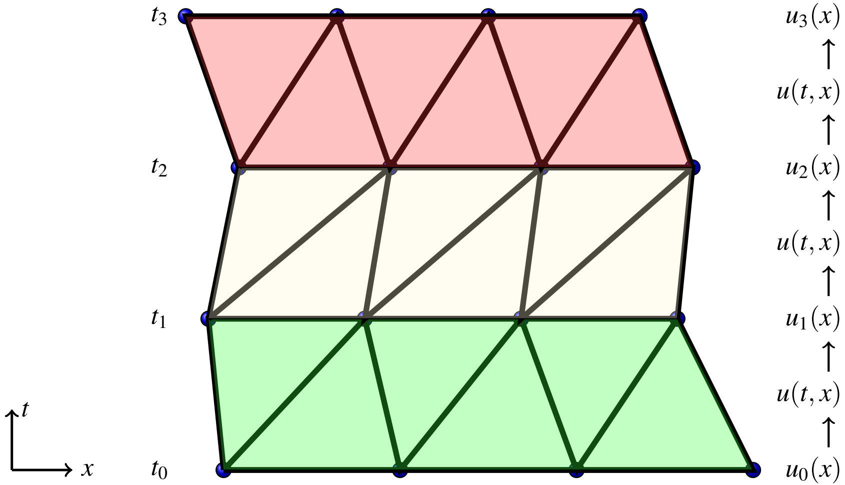
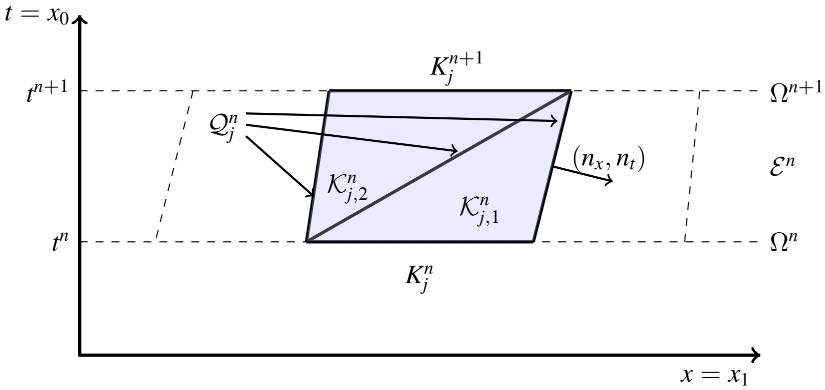
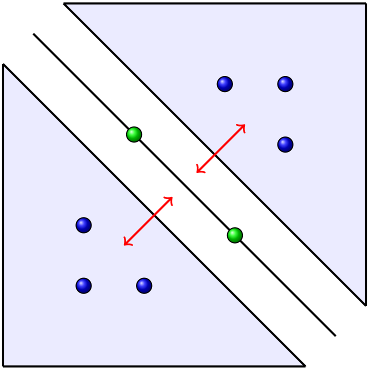


Space-time notations - simplices



HDG



$$u_h \in V_h = \left\{ v_h \in L^2(\Omega),\ v_h \in P^k(K)\ \forall K \in \mathcal{T}_h \right\}$$

$$\overline{u}_h \in \overline{V}_h = \left\{ \overline{v}_h \in L^2(\mathcal{F}),\ \overline{v}_h \in P^k(F)\ \forall F \in \mathcal{F} \right\}$$

No facet unknowns on time levels

Fluid-rigid body interactions

Time dependent Navier-Stokes

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p - 2\nu \nabla \cdot \varepsilon(u) &= f && \text{in } \Omega(t) \\ \nabla \cdot u &= 0 && \text{in } \Omega(t) \end{aligned}$$

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ST-EHDG

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ST-EHDG

Vertical displacement and rotation

$$\begin{aligned} m\ddot{d} + c_y\dot{d} + k_y d &= F_y, \\ I_\theta \ddot{\theta} + c_\theta \dot{\theta} + k_\theta \theta &= M, \end{aligned}$$

where F_y is the lifting force, M is the pitching moment force

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Predictor-corrector (BDF2)

Discretization

Viscous term: hybridized IP-DG discretization

Pressure terms: standard hybridization

Convection: nonlinear upwinding

In space-time slab n

Find $(u_h, \bar{u}_h, p_h, \bar{p}_h) \in V_h \times \bar{V}_h \times Q_h \times \bar{Q}_h$ such that

$$t_h^n(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + a_h^n(\mathbf{u}_h, \mathbf{v}_h) + b_h^n(\mathbf{p}_h, \mathbf{v}_h) - b_h^n(\mathbf{q}_h, \mathbf{u}_h) = \sum_{\mathcal{K} \in \mathcal{T}^n} \int_{\mathcal{K}} f \cdot v_h \,\mathrm{d}x - \int_{\partial \mathcal{E}^N \cap I_n} g \cdot \bar{v}_h \,\mathrm{d}s + \int_{\Omega_n} u_h^- \cdot v_h \,\mathrm{d}s$$

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Nonlinearity - Picard Iteration

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Nonlinearity - Picard Iteration

Linear system - Static condensation

Static Condensation

Using U and $\overline{U}$ for the coefficient vectors

Block system

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U \\ \overline{U} \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}$$

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A is block diagonal

$$U = A^{-1}(F - B\overline{U})$$
$$CA^{-1}(F - B\bar{U}) + D\overline{U} = G$$

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Final problem

$$\begin{aligned} (D - CA^{-1}B)\overline{U} &= G - CA^{-1}F \\ U &= A^{-1}(F - B\overline{U}) \end{aligned}$$

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Typically, the number of DoFs is smaller for HDG than for DG

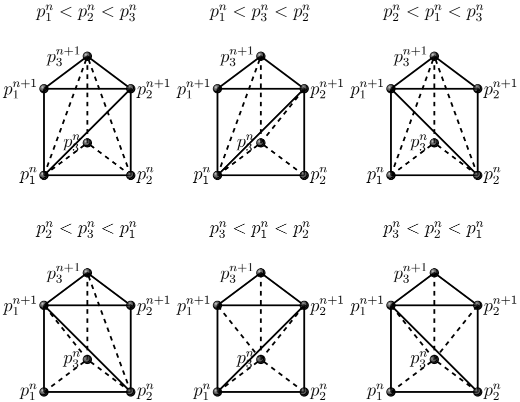
Comparison (on simplices - moving domain)

	ST-HDG*	ST-EDG†	ST-EHDG†
div-free velocity	✓	✓	✓
div-conforming velocity	✓	×	✓
energy-stable	✓	✓	✓
loc. mom. conserving	✓	×	✓
number of degrees-of-freedom	largest	smallest	significantly < ST-HDG slightly > ST-EDG

*T.L. Horvath and S. Rhebergen, A locally conservative and energy-stable finite element method for the Navier–Stokes problem on time-dependent domains, Int. J. Numer. Meth. Fluids, 89/12 (2019), pp 519-532.

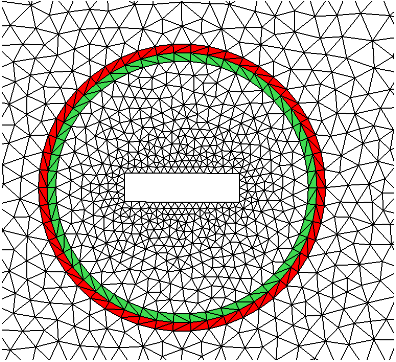
†T.L. Horvath and S. Rhebergen, An exactly mass conserving space-time embedded-hybridized discontinuous Galerkin method for the Navier-Stokes equations on moving domains, J. Comp. Phys. 417 (2020)

Basic meshing idea



Use the diagonal cut from the node with the smallest index (staircase triangulation)

Sliding mesh generation



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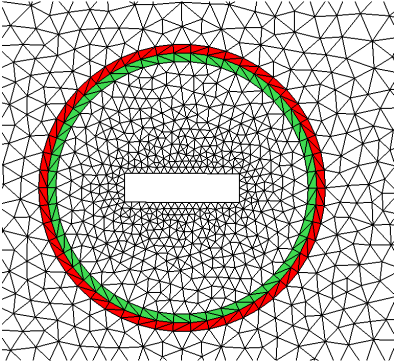
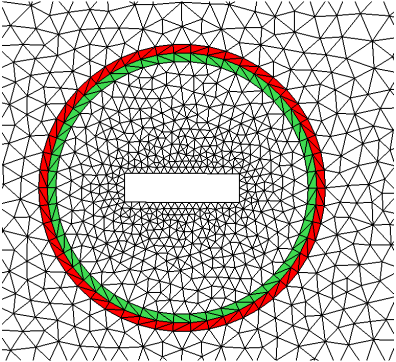


Inner region rotates with the object

Sliding mesh generation



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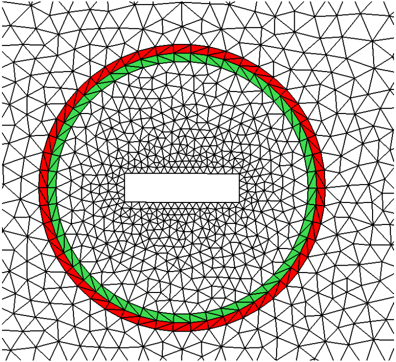

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In between: sliding layer

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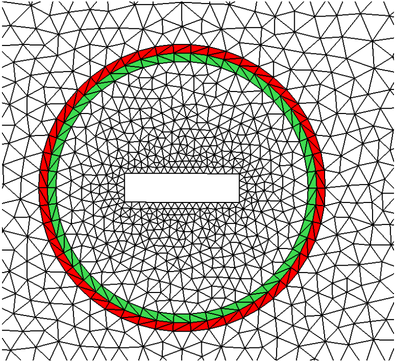

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In between: sliding layer

Buffer layer allows precomputable meshes

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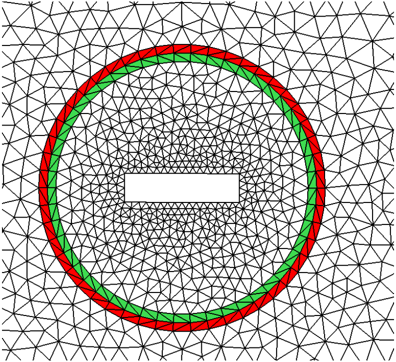

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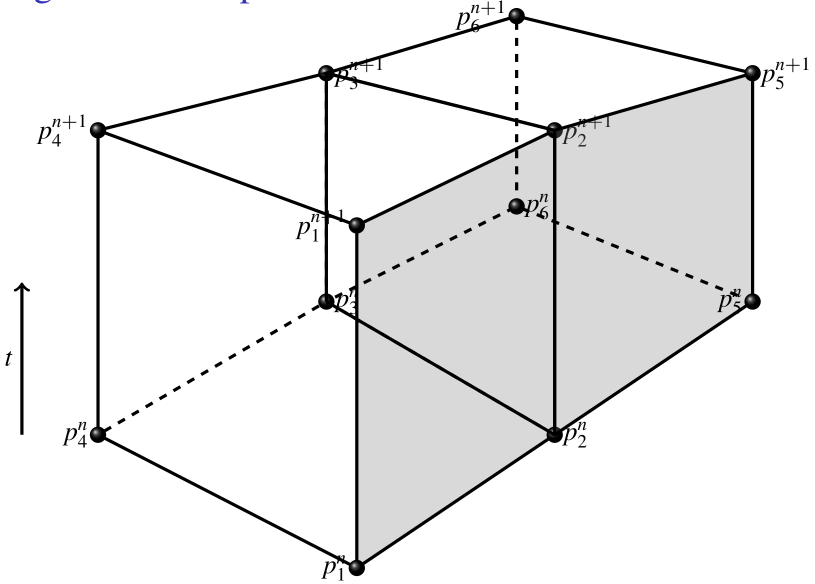
"Build your own mesh"[a]

[a] T.L. Horvath and S. Rhebergen, A conforming sliding mesh technique for an embedded-hybridized discontinuous Galerkin discretization for fluid-rigid body interaction (2022), Int. J. Numer. Meth. Fluids, 94/11 (2022), pp. 1784-1809

Sliding block example



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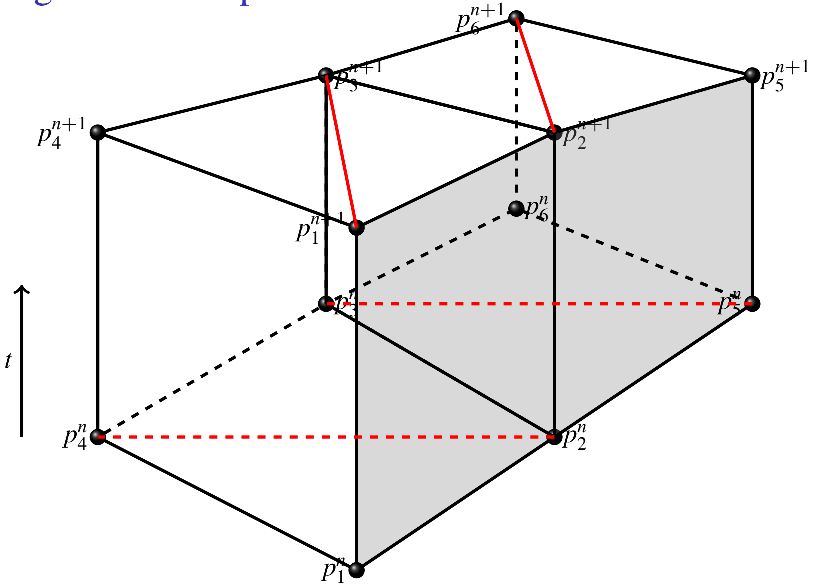

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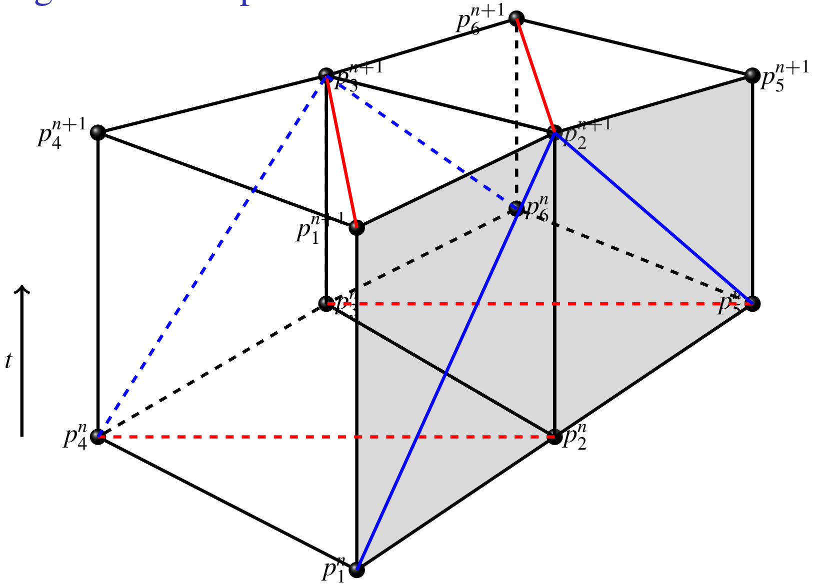

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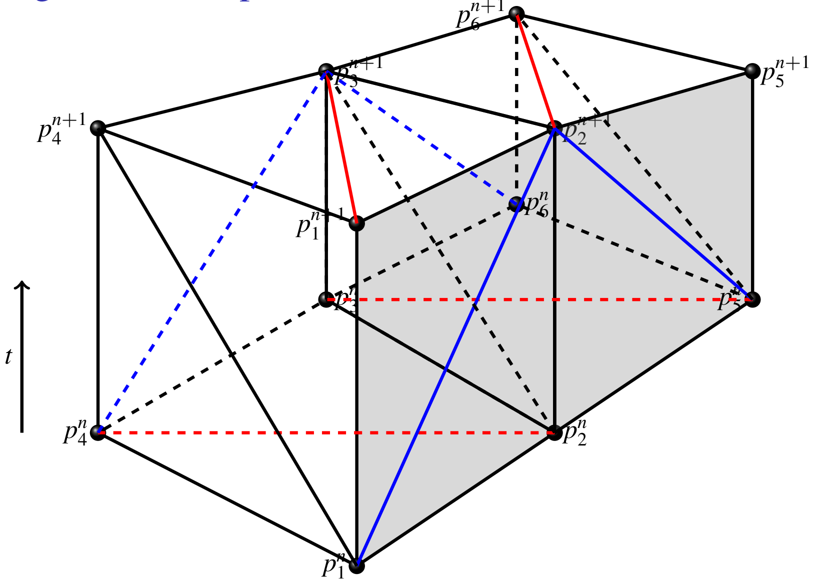
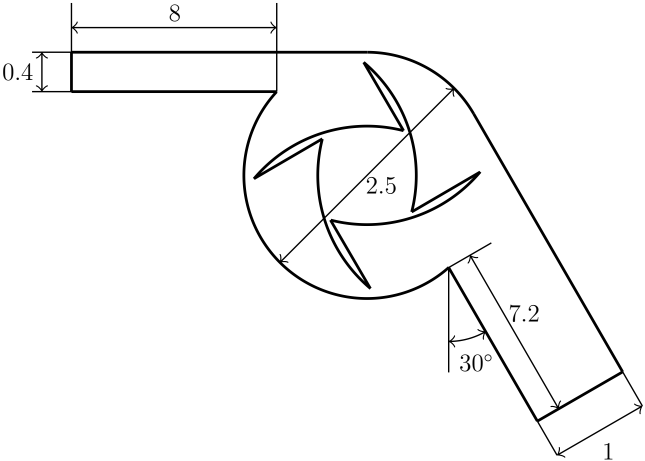


Rotating turbine

- Parabolic inflow, $\max |u_{in}| = 50$



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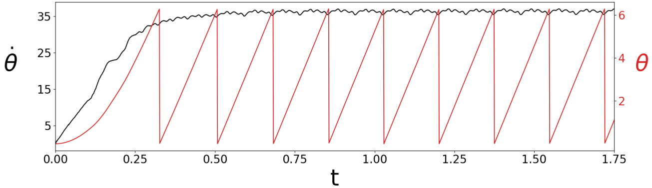

Adaptive mesh refinement

AMR with Derefinement (coarsening)

Only refine the mesh where it is needed

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Refinement only for time-dependent problems can be too expensive

Adaptive mesh refinement

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Refinement only for time-dependent problems can be too expensive

It needs to be combined with derefinement (coarsening)

Space-time adaptivity[‡]

Space and time adaptivity

We can "eliminate" or create new spatial elements

[‡] **TLH, E. Smeu** A conforming space-time mesh adaptation with local coarsening, in preparation

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Slab-by-slab approach: simplices allow local time refinement

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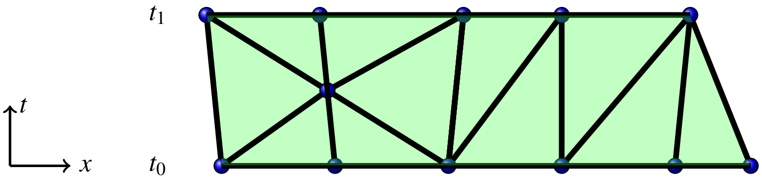

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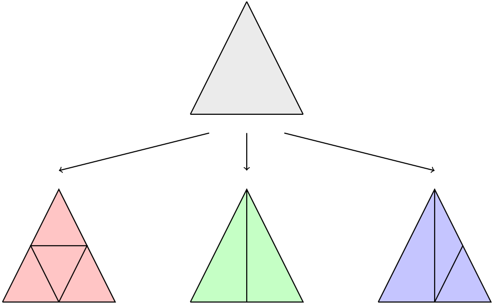
2D RGB refinement[§]

[§] Funken, S. A., Schmidt, A. (2021). A coarsening algorithm on adaptive red-green-blue refined meshes. Numerical Algorithms, 87, 1147-1176.

2D RGB refinement[§]



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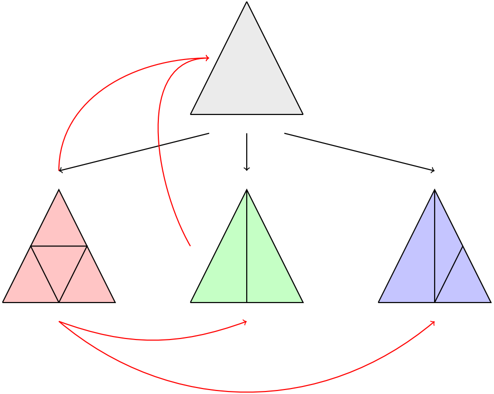

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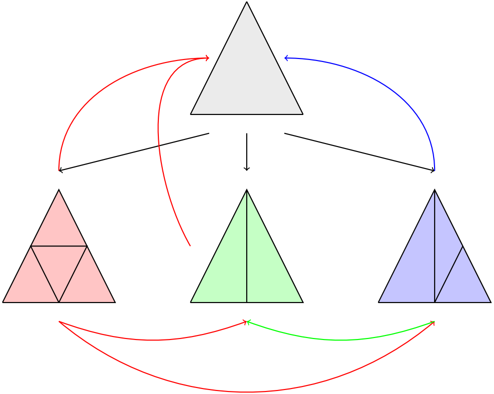
2D RGB refinement§



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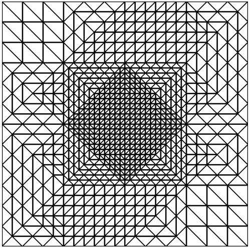

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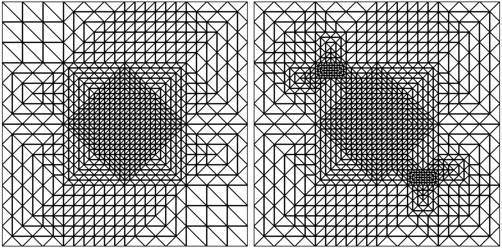

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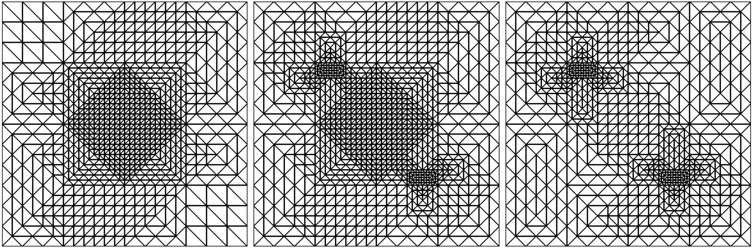
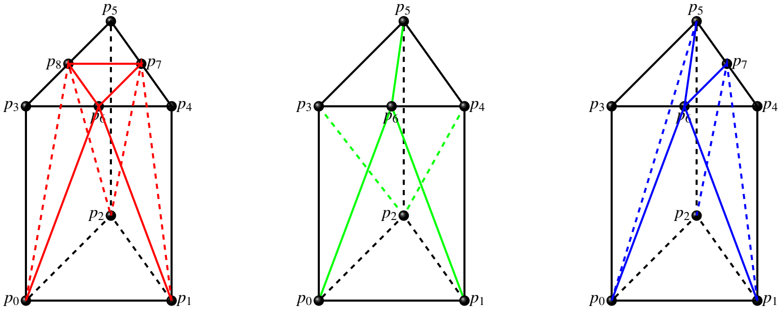
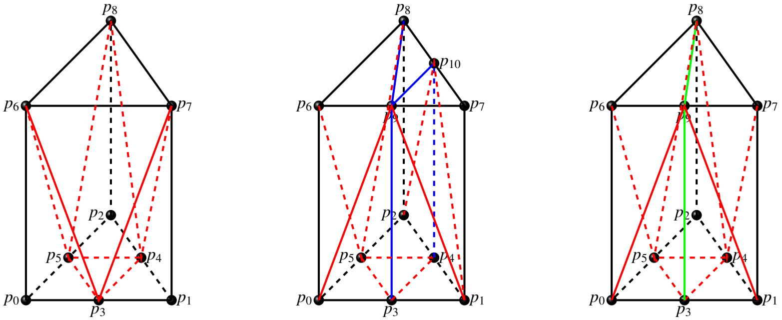


Extension to ST I - no need of projection[¶]



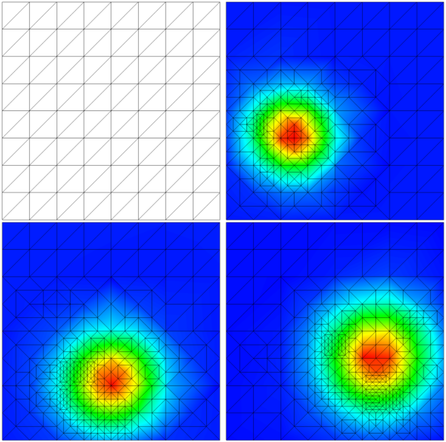
¶ 9 different space-time elements

Extension to ST II - no need of projection‖

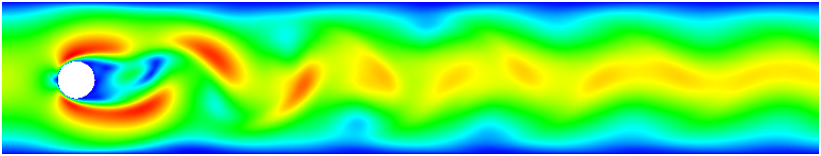


‖8 different space-time elements

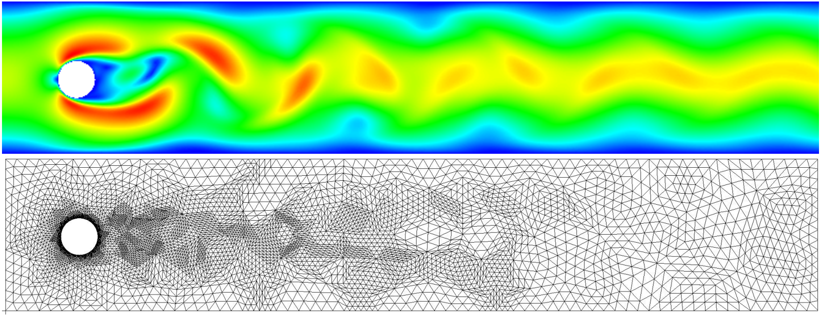
Rotation Gaussian pulse - advection-diffusion



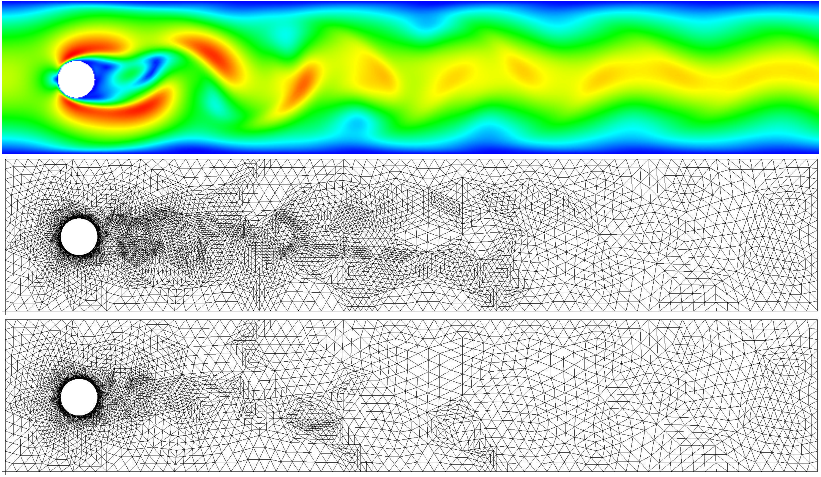
Flow past the cylinder example



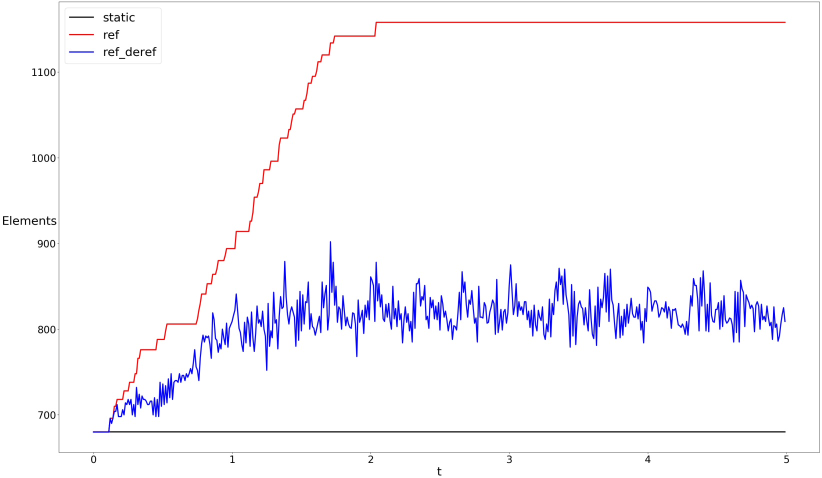
Flow past the cylinder example



Flow past the cylinder example

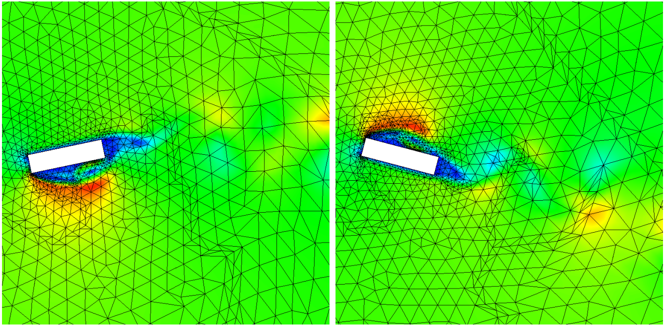


Number of mesh elements



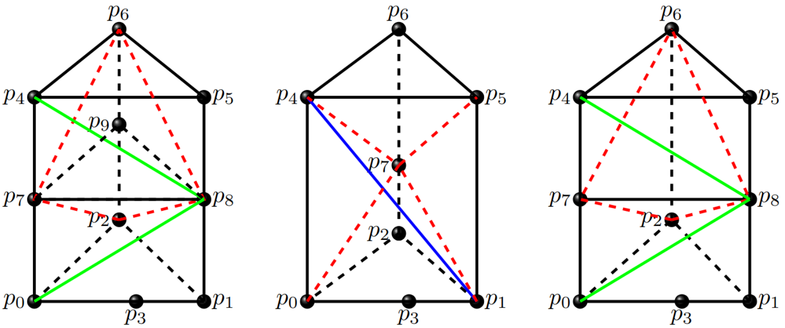
FRBI with mesh adaptivity

Rotational galloping

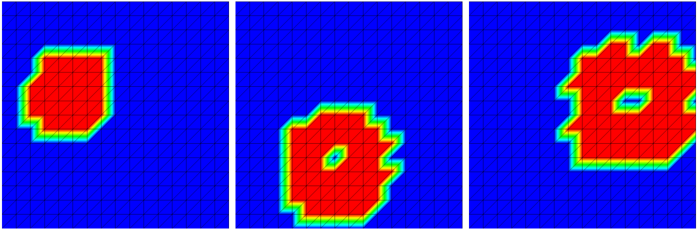


Maximum angle and frequency of the oscillation is better matching the literature then in our previous work while using fewer elements

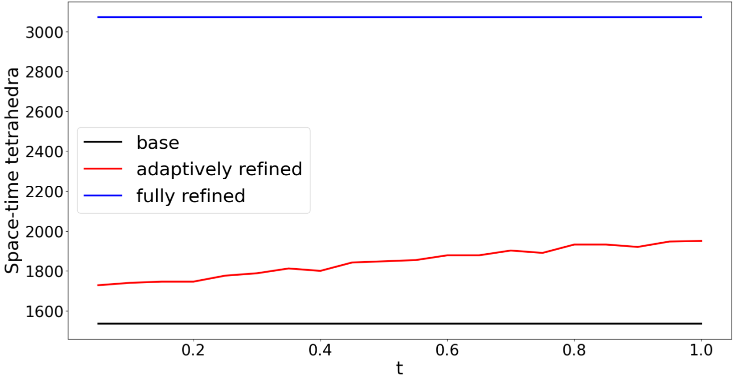
Time adaptivity



Rotation Gaussian pulse - advection-diffusion



Time adaptivity



mesh	unrefined	fully refined	adaptively refined
L^2 error	$8.99 \cdot 10^{-5}$	$6.2 \cdot 10^{-5}$	$6.2 \cdot 10^{-5}$

Extension to 4D

MFEM 4D branch

Extension to 4D

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Base case: tetrahedra to 4D space-time "wedge" to 4D pentatope

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The diagonal from the node with the lowest index works (staircase triangulation) - 24 options

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AMR with coarsening without mesh history: not with RGB

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AMR with coarsening without mesh history: not with RGB

Predecessor of the RGB paper: an NVB paper, might generalize to 3D

Conclusion & Future work

Conclusion

- Refinement and coarsening without projecting the solution
- Local temporal refinement

Future work

- Time refinment next to the obtacle (high order ODE solver)
- Extend to 3+1 D (currently working on it)
- Fluid-Structure Interaction
- Cahn-Hilliard equation (phase separations: sharp moving discontinuities)

https://thorvath12.github.io

Thank you!

Thank you!

Midwest Numerical Analysis Day 2027
May 21-22, 2027 at Oakland University